

# Energy fluctuation of ideal Fermi gas trapped under generic power law potential $U = \sum_{i=1}^d c_i \left| \frac{x_i}{a_i} \right|^{n_i}$ in d dimension

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## Abstract

Energy fluctuation of ideal Fermi gas trapped under generic power law potential  $U = \sum_{i=1}^d c_i \left| \frac{x_i}{a_i} \right|^{n_i}$  have been calculated in arbitrary dimension. Energy fluctuation is scrutinized further in the degenerate limit  $\mu \gg K_B T$  with the help of Sommerfeld expansion. The dependence of energy fluctuation on dimensionality and power law potential is studied in detail. Most importantly our general result can exactly reproduce the recently published result regarding free and harmonically trapped ideal Fermi gas in d=3[1].

## 1 Introduction

A lot of theoretical studies[2, 3, 4, 5, 6, 7] have been done on the subject of ideal free quantum gases even before the experimental observation of Bose-Einstein condensation (BEC) and Fermi degeneracy. But, this subject drew more attention after it had been possible to detect BEC[8, 9, 10] and Fermi degeneracy[11] experimentally in trapped quantum gases. Since then, an increasing attraction is noticed in the subject of trapped quantum gases. Although a lot of theoretical studies[1, 12, 13, 14, 15, 16, 17, 18, 19] are done on quantum gases trapped under generic power law potential,  $U = \sum_{i=1}^d c_i \left| \frac{x_i}{a_i} \right|^{n_i}$ , none of these contained detailed discussion on energy fluctuation  $\Delta\epsilon^2$ , until the recent paper of Biswas et. al.[1] where they discussed the energy fluctuation for free and harmonically trapped quantum gases in three dimensional space. In their paper they have also conjectured a relation between discontinuity of  $C_V$  and energy fluctuation  $\Delta\epsilon^2$ , suggesting the appearance of a hump in  $\frac{\Delta\epsilon^2}{kT^2}$  over its classical limit does indicate a discontinuity of  $C_V$ . It was also reported in their study that, there is no hump in  $\frac{\Delta\epsilon^2}{kT^2}$ , in the case of free and harmonically trapped ideal Fermi gases. In their recent paper, Mehedi et. al. has proved the conjecture and investigated energy fluctuation[20] in details for ideal Bose gas trapped under generic power law potential in arbitrary dimension. Point to note that, Biswas et. al. calculated the  $\Delta\epsilon^2$  for three dimensional free and harmonically trapped quantum gases, but  $\Delta\epsilon^2$  is still not examined in arbitrary dimension while Fermi gas is trapped under generic power law potential.

In this paper we investigate the  $\Delta\epsilon^2$  of ideal Fermi gases trapped under generic power law potential  $U = \sum_{i=1}^d c_i |\frac{x_i}{a_i}|^{n_i}$  in  $d$  dimension. At first we determine the density of states, which enables us to calculate the energy fluctuation using the Fermi distribution function. Later, we scrutinize the energy fluctuation in the quantum degenerate limit using the Sommerfeld expansion. The dependence of energy fluctuation on dimensionality and power law exponents are visited in detail. Interestingly our more general final result of energy fluctuation can exactly reproduce the same result in three dimension, for free and harmonically trapped Fermi gases reported in Biswas et. al. [1] by choosing  $d = 3$ ,  $n = \infty$  (free Fermi gas in three dimension) and  $d = 3$ ,  $n = 2$  (harmonically trapped Fermi gas in three dimension).

## 2 Energy fluctuation of trapped Fermi gas

Considering an ideal gas trapped under a generic power law potential in  $d$  dimensional space with a single particle hamiltonian[13],

$$\epsilon(p, x_i) = bp^l + \sum_{i=1}^d c_i |\frac{x_i}{a_i}|^{n_i} \quad (1)$$

where,  $p$  is the momentum and  $x_i$  is the  $i$  th component of coordinate of a particle and  $b, l, a_i, c_i, n_i$  are defined as all positive constants. Here,  $c_i, a_i, n_i$  determine the depth and confinement power of the potential and  $l$  being the kinematic parameter, where  $x_i < a_i$ . As  $|\frac{x_i}{a_i}| < 1$ , the potential term goes to zero as all  $n_i \rightarrow \infty$ . We can construct our usual non-relativistic Hamiltonian with  $l = 2$  and  $b = \frac{1}{2m}$ . The density of states for such system is [15, 19],

$$\rho(\epsilon) = C(m, V'_d) \epsilon^{\chi-1} \quad (2)$$

where,  $C(m, V'_d)$  is a constant depending on effective volume  $V'_d$ [14, 16]<sup>1</sup> and  $\chi = \frac{d}{l} + \sum_i \frac{1}{n_i}$ . Now the Fermi distribution function, is given by

$$\bar{n}_i = \frac{1}{z^{-1} e^{\beta\epsilon_i} + 1} \quad (3)$$

where,  $z$  is fugacity. So, the energy fluctuation of trapped fermi gas,

$$\begin{aligned} \Delta\epsilon^2 &= \bar{\epsilon}^2 - \bar{\epsilon}^2 = \sum_i \bar{n}_i \epsilon_i^2 - (\sum_i \bar{n}_i \epsilon_i)^2 = \int d\epsilon \rho(\epsilon) \epsilon^2 n(\epsilon) - (\int d\epsilon \rho(\epsilon) \epsilon n(\epsilon))^2 \\ &= (kT)^2 [\chi(\chi+1) \frac{f_{\chi+2}(\sigma)}{f_{\chi}(\sigma)} - \chi^2 \frac{f_{\chi+1}^2(\sigma)}{f_{\chi}^2(\sigma)}] \end{aligned} \quad (4)$$

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<sup>1</sup>to read a detail discussion on effective volume, see [13, 14, 15, 16]

where,  $f_p(z)$  is the Fermi function defined as,

$$f_p(z) = \int_0^\infty dx \frac{x^{p-1}}{z^{-1}e^x + 1} = \sum_{j=1}^\infty (-1)^{j-1} \frac{z^j}{j^p} \quad (5)$$

### 3 Energy fluctuation of trapped Fermi gas in the degenerate limit

At low temperature, we can approximate the Fermi function and write it as quickly convergent Sommerfeld series[4]

$$f_p(z) = \frac{(\ln z)^p}{\Gamma(p+1)} [1 + p(p-1) \frac{\pi^2}{6} \frac{1}{(\ln z)^2} + p(p-1)(p-2)(p-3) \frac{7\pi^4}{360} \frac{1}{(\ln z)^4} + \dots] \quad (6)$$

From Ref.[19] we can write the chemical potential (fugacity) as below,

$$\mu = kT \ln z = E_F [1 - (\chi - 1) \frac{\pi^2}{6} (\frac{kT}{E_F})^2] \quad (7)$$

The expression of Fermi energy for Fermi gas trapped under generic power law potential can be found in Ref. [19]. So, using the Sommerfeld approximation we can re-write the energy fluctuation from eq. (4),

$$\Delta\epsilon^2 = \frac{\chi}{(\chi+2)(\chi+1)} (kT \ln z)^2 + \frac{\pi^2}{3} \frac{\chi(2\chi+1)}{\chi+2} (kT)^2 - \frac{2\pi^2}{3} \frac{\chi^2}{(\chi+1)^2} (kT)^2 \quad (8)$$

Again using of Eq. (7), the energy fluctuation becomes

$$\frac{\Delta\epsilon^2}{E_F^2} = \frac{\chi}{(1+\chi)^2(2+\chi)} + \frac{1}{3} \frac{1}{(1+\chi)^2} \pi^2 \tau^2 + \frac{1+34\chi+40\chi^2-8\chi^3-13\chi^4+2\chi^6}{36(1+\chi)^2(2+\chi)} \pi^4 \tau^4 \quad (9)$$

where,  $\tau = \frac{T}{T_F}$ . At  $T = 0$ , the energy fluctuation becomes,

$$\Delta\epsilon_0^2 = \frac{\chi}{(\chi+2)(\chi+1)} E_F^2 \quad (10)$$

In the case of ideal free Fermi gas in three dimensional space,  $\chi = 3/2$ . So, from Eq. (4) we see  $\Delta\epsilon^2$  becomes,

$$\frac{\Delta\epsilon_0^2}{E_F^2} = \frac{12}{175} + \frac{\pi^2}{5} \tau^2 + \frac{329\pi^4}{2400} \tau^4 \quad (11)$$

And when the ideal Fermi gas is trapped under a harmonic potential in three dimension, ( $\chi = 3$ )

$$\frac{\Delta\epsilon_0^2}{E_F^2} = \frac{3}{80} + \frac{\pi^2}{4} \tau^2 + \frac{163\pi^4}{240} \tau^4 \quad (12)$$

Equation (11) and (12) coincides exactly with Biswas et. al. [1]

## 4 Results and Discussion

In this section we summarize the interesting findings relating energy fluctuation of ideal Fermi gas trapped under generic power law potential.

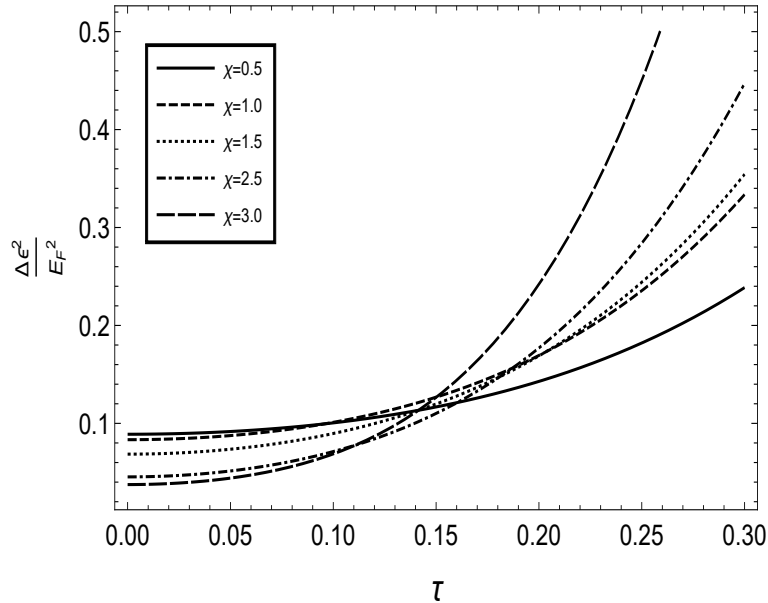


Figure 1: Energy fluctuation ideal trapped Fermi gas as a function of  $\tau = \frac{T}{T_F}$ , with different power law potentials.

It is seen in the studies that[5], all the thermodynamic quantities of free Fermi (Bose) gases can be presented in terms of Fermi function (Bose function) depending on dimensionality  $d$ . Now, the thermodynamic quantities of trapped Fermi (Bose) gases can still be written in terms of Fermi function (Bose function), using the concept of effective volume and effective thermal wavelength[13, 19]. But in this case the Fermi (Bose) functions depend on  $\chi = \frac{d}{l} + \sum_i^d \frac{1}{n_i}$ . So, at first we explore the dependence of energy fluctuation on  $\tau = \frac{T}{T_F}$ , with varying  $\chi$  (figure 1). Here, all  $n_i \rightarrow \infty$  correspond to free system[13]. It is very enthralling to point out that, the energy fluctuation is non-zero at  $T = 0K$  for any value of  $\chi$ , unlike the Bose gas for which the energy fluctuation is zero at  $T = 0K$  for any value of  $\chi$ [1, 20]. But, interestingly, it is clear from the figure 1 that, the value of energy fluctuation at  $T = 0K$ , changes with varying  $\chi$ . We will explore this phenomena in detail, later. But, no hump is noted in the energy fluctuation of Fermi gas, unlike the Bose gas. This result is in agreement with Biswas et. al.[1].

It has already been reported, within the canonical ensemble that, energy fluctuation  $\Delta\epsilon^2$  is related to specific heat  $C_V$  as  $\Delta\epsilon^2 = kT^2C_V$ . And in the case of grand canonical

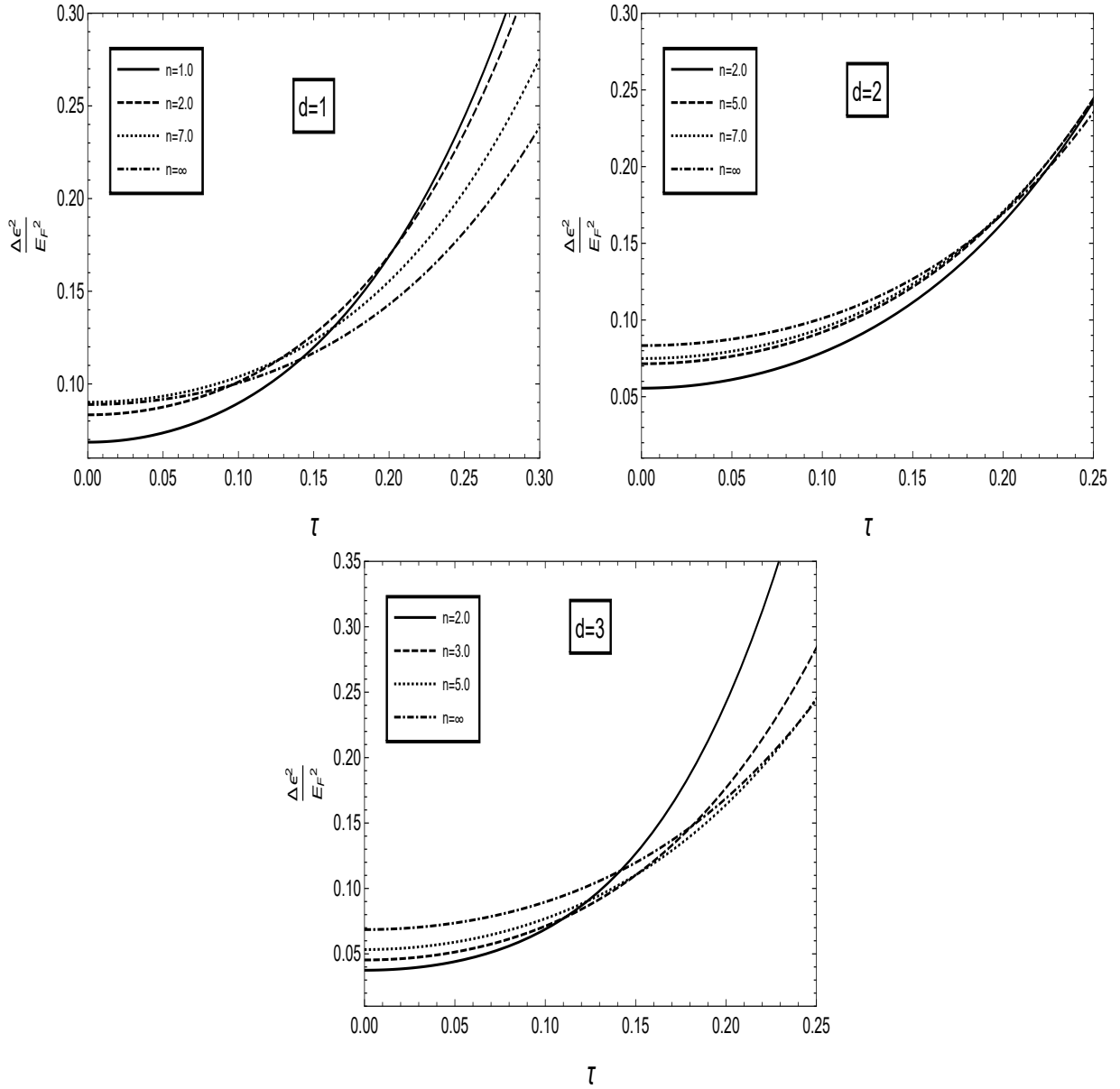


Figure 2: Energy fluctuation of ideal trapped Fermi gas in  $d = 1$ ,  $d = 2$  and  $d = 3$ .

ensemble this is true for ideal classical gas[1] only. Ref. [1, 20], shows how the non zero fugacity of quantum gas causes this relation to remain invalid for quantum system. Now the quantum gases behave as a classical gas in the high temperature limit and thus tend to maintain this relation at high temperature. So, the status of this relation to be invalid is very important in low temperature limit. The low temperature limit of Bose gas corresponds to condensed phase. And the status of this relation has already been checked for Bose system by Mehedi et. al. [20]. And it can be checked for trapped Fermi gas in the degenerate limit with the help of Eq. (9), which depicts the energy fluctuation of trapped fermi gas changes as  $\Delta\epsilon^2 \sim A_1 + A_2T^2 - A_3T^4$ , while  $C_V$  changes as  $C_V \sim AT$ [20], where  $A, A_i$  are functions of  $\chi$ . The temperature dependency of  $\Delta\epsilon^2$  and  $C_V$  explicitly shows, how  $\Delta\epsilon^2 = kT^2C_V$  relation is not maintained

in low temperature limit of trapped Fermi gas.

Let us further analyse the energy fluctuation in different space dimensions. In figure 2 we have set the condition of symmetric potential i.e.  $n_1 = n_2 = \dots = n_d = n$ . The non zero energy fluctuation at  $T = 0K$  is visible in all the figures. But this non zero value of energy fluctuation at  $T = 0K$ ,  $\Delta\epsilon_0^2$  changes with different trapping potential. To be more specific,  $\Delta\epsilon_0^2$  decreases with decreasing value of  $n$ . When all  $n_i = \infty$  (free system)  $\Delta\epsilon_0^2$  has the highest value and this value reduces as we decrease  $n$  (figure 2). One can also find out from figure 2 (a)-(c) that,  $\Delta\epsilon_0^2$  also changes with dimensionality. In order to see this in detail we have done a separate plot. Nevertheless, the influence of trapping potential is observed not only at  $T = 0K$  but also in the whole temperature range for any dimensionality.

In figure 3, we analyse the change of  $\Delta\epsilon_0^2$  with respect to dimensionality, for dif-

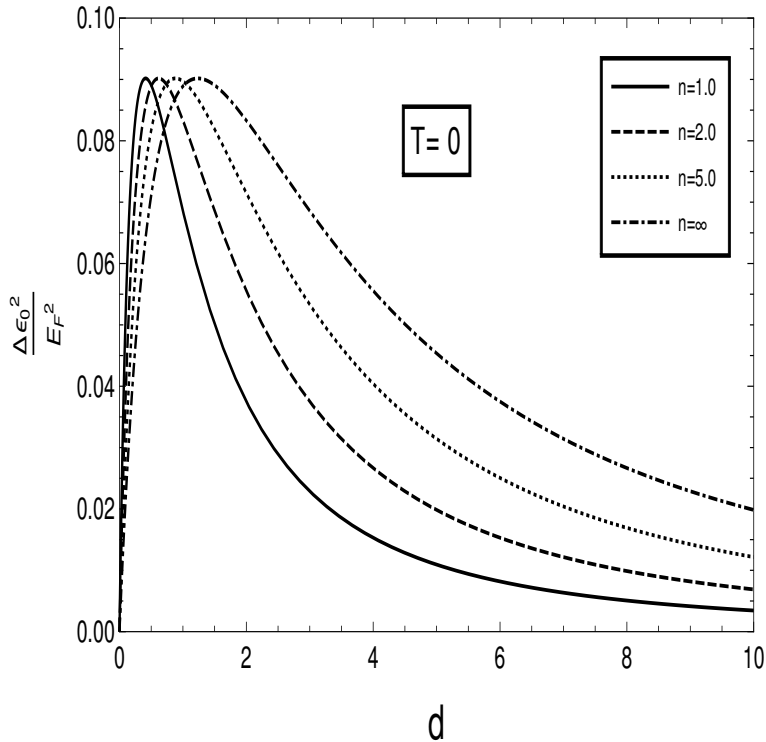


Figure 3: Change of energy fluctuation at  $T = 0K$  ( $\Delta\epsilon_0^2$ ) with respect to dimensionality.

ferent types of trapping potential. It is seen from the figure that, for free Fermi system the  $\Delta\epsilon_0^2$  is maximum near  $d = 1$ . But this situation changes while the Fermi system is trapped, as a shift is noticed in the maximum of  $\Delta\epsilon_0^2$ . Another important point to notice from figure 3 that, the maximum of  $\Delta\epsilon_0^2$  can be obtained in different dimensionality (depends on the trapping potential), but the maximum value of  $\Delta\epsilon_0^2$  remains the same for all. This figure is very significant, as from it one can predict which trapping potential will cause maximum of energy fluctuation at  $T = 0K$  for any

specific dimensionality.

## 5 Conclusion

In this manuscript we have restricted our discussion for ideal nonrelativistic fermi gases. Our general result on energy fluctuation of ideal Fermi gas trapped under generic power law potential in arbitrary dimension can reproduce exactly the same result in three dimension[1] but, it will be interesting to calculate energy fluctuation for interacting fermions, which is not yet done. Interaction might change the dependence of energy fluctuation on trapping potential. However, we are currently investigating the energy fluctuation for ideal relativistic Fermi gases by taking into account the presence of antiparticles.

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